NRCPS-HE-36-2010

Large Spin Behavior of Anomalous Dimensions

and

Short-long Strings Duality

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Abstract

We are considering the semi-classical string soliton solution of Gubser, Klebanov and Polyakov which represents highly excited states on the leading Regge trajectory, with large spin in AdS_5. A prescription relates this soliton solution with the corresponding field theory operators with many covariant derivatives, whose anomalous scaling dimension grows logarithmically with the space-time spin. We develop an iteration procedure which, in principle, allows to derive all terms in the large spin expansion of the anomalous scaling dimension of twist two operators at strong coupling. We explicitly derive the dependence of anomalous dimension on spin for all leading and next-to-leading orders. Our string theory results are consistent with the conjectured "reciprocity" relation, which has been verified to hold in perturbation theory up to five loops in $N = 4$ SYM. We also derive a duality relation between long and short strings.

1 *Introduction*

It was found in [1] that the type IIB string theory soliton solution with large spin on $AdS_5 \times S^5$ describes the gauge theory operators with many covariant derivatives, whose anomalous scaling dimension grows logarithmically with the Lorentz spin S. Operators with many covariant derivatives are present in QCD where they were studied in the context of deep inelastic scattering and have the following general structure [4, 5, 6]

$$Tr\{\Phi\nabla_{\lambda_1}...\nabla_{\lambda_S}\Phi\}.$$

In free field theory such operators have dimension $\Delta = S + 2$, while their space-time spin is S. In the interacting gauge field theories the scaling is violated and the anomalous dimension is different from zero,

$$\gamma(\lambda, S) = \Delta - (S+2) = f(\lambda)\ln S + f_0(\lambda) + f_{11}(\lambda)\frac{\ln S}{S} + f_1(\lambda)\frac{1}{S} + ... \tag{1}$$

Anomalous dimension of these high spin operators is a function of the coupling constant $\lambda^2 = g_{YM}^2 N/4\pi$ and of the spin S and can be computed order by order in the perturbation theory. The weak coupling corrections are dominated by the leading logarithmic term $f(\lambda)\ln S$, where $f(\lambda)$ is the so called "cusp anomalous dimension".

The AdS/CFT correspondence [17, 18, 19] allows the strong coupling description of the same physical quantity, identifying the energy of the string with the conformal dimension $E = \Delta$ of the dual operator. Thus the dual, string theory description, provides the strong coupling behavior of the anomalous dimensions and it appears also to grow logarithmically with spin [1, 2, 3]. In this article we are intersected in the behavior of subleading in S terms of the anomalous dimension $\gamma(\lambda, S)$ in the weak and strong coupling regimes.

We have found that the anomalous dimension in the strong coupling regime is given by the following expression

$$\gamma(\lambda, S) = f\ln(S/\sqrt{\lambda}) + f_0 + \sum_{n=1}^{\infty} f_{nn}\frac{\ln^n(S/\sqrt{\lambda})}{S^n} + \sum_{n=1}^{\infty} f_{nn-1}\frac{\ln^n(S/\sqrt{\lambda})}{S^{n+1}} + f_1\frac{1}{S} + f_2\frac{1}{S^2} + ... \tag{2}$$

where

$$\begin{aligned}
f &= \frac{\sqrt{\lambda}}{\pi} \\
f_0 &= \frac{\sqrt{\lambda}}{\pi}\,(2\rho_0 + \ln(\pi/2)) = \frac{\sqrt{\lambda}}{\pi}\,(\ln 8\pi - 1) \\
f_1 &= \frac{4\lambda}{\pi^2}\,(\rho_1 + \ \rho_{11}\ln(\pi/2)) = \frac{\lambda}{2\pi^2}\,(\ln 8\pi - 1) \\
f_2 &= \frac{\lambda^{3/2}}{8\pi^3}(-\ln^2 8\pi + \frac{9}{2}\ln 8\pi - 4\ln 2 - 1) \\
f_{nn} &= (-1)^{n+1}\frac{\lambda^{\frac{n+1}{2}}}{(2\pi)^{n+1}}\frac{2}{n} = \frac{(-1)^{n+1}}{2^n n}\,f^{n+1}, \qquad n = 1, 2, ... \\
f_{nn-1} &= (-1)^{n+1}\frac{\lambda^{\frac{n+2}{2}}}{(2\pi)^{n+2}}\,(\ \frac{n+4}{2} + 2\sum_{k=1}^{n}\frac{1}{k} - 2\ln 8\pi\ ), \qquad n = 1, 2, ...
\end{aligned} \tag{3}$$

for

$$\sqrt{\lambda} \gg 1, \qquad \frac{S}{\sqrt{\lambda}} \gg 1. \tag{4}$$

Two comments are in order. Firstly, notice the appearance of the harmonic sum in the coefficients of the subleading logarithms f_{nn-1} [1]. Secondly, we should mention that the structure of the large spin expansion at strong coupling (2) is the same with the large spin expansion in perturbation theory [23].

Using the so-called BES equation [8] the cusp anomalous dimension $f(\lambda)$ was thoroughly studied both in the weak [8, 11] and strong coupling regime [9, 10, 11]. Furthermore, all contributions to the exact in S anomalous dimension (2) which are free from wrapping effects can, in principle, be calculated [12, 15] by using a linear integral equation previously derived in [14] [2]. Finally, the solution of the thermodynamic Bethe Ansatz (TBA) of [16] can, in principle, give the exact result including wrapping effects. Since our strong coupling coefficients (3) include all effects due to wrapping interactions it would be interesting to solve the TBA equations at strong coupling and compare with our results.

The formulae (3) allow to check the reciprocity relation which assumes the following functional relation for γ

$$\gamma(S) = P(S + \frac{1}{2}\gamma(S)), \tag{5}$$

where $P(S)$ satisfies a "parity preserving" or "reciprocity" relation [25, 26] which can be cast in the form [27]

$$P(S) = \sum_{k=0}^{\infty} \frac{c_k(\ln C)}{C^{2k}}. \tag{6}$$

In (6), C is the "bare" quadratic Casimir operator of the $SL(2,R)$ group given by $C = S(S+1)$. The reciprocity relation (6) has been verified to hold in perturbation theory up to three loops in QCD [27] and up to five loops in $N = 4$ SYM [22, 28, 29, 30, 32, 35]. It was also found to hold in string perturbation theory at the classical level in [23] and at one-loop up to order $\frac{1}{S^3}$ in [23, 24].

Equivalently, the first two relations following from (6)

$$\begin{aligned} f_1 &= \frac{1}{2} f(f_0 + 1) \\ f_{21} &= \frac{1}{16} f[f^3 - 2f^2(f_0+1) - 16 f_{10}] \end{aligned} \tag{7}$$

were verified [23, 31]. Since now we have the infinite series of the coefficients of the subleading logs in the strong coupling expansion (2) it is, in principle, possible to check if the relations following from the reciprocity property of the function P hold at any level. By isolating the appropriate terms we get the next relation in the sequence (7). This reads

$$\begin{aligned} f_{43} + [f_{22}^2 + 2f_{11}f_{33} + 2ff_{32} + f_{44}(2f_0 - \frac{5}{2}f)] + f[3f_{11}f_{22} + \frac{3}{2}ff_{21} + f_{33}(-\frac{35}{8}f + 3f_0)] \\ +f^2[\frac{3}{4}f_{11}^2 + \frac{1}{2}ff_{10} + f_{22}(-\frac{65}{24}f + \frac{3}{2}f_0)] + f^3[\frac{1}{16}ff_1 + f_{11}(-\frac{125}{12\ 16}f + \frac{1}{4}f_0)] \\ +f^5[\frac{25}{12\ 32}f - \frac{1}{32}f_0] = 0. \end{aligned} \tag{8}$$

[1] Our f_{nn-1}'s are related to the ones appearing in [23] by $f_{nn-1}^{(ours)} = f_{n+1n}^{([23])}$.

[2] When the length of the operator is $L > 3$ there are additional terms in the large S expansion of the anomalous dimension which scale like $1/\ln^n S$ [13].

Now we can use (3) to find and substitute the values for the various coefficients needed in (8) to find that this equation is, indeed, satisfied. Thus, we have seen that the constraints coming from the reciprocity relation (6) are consistent with the infinite series of coefficients of (3).

In the next section we shall derive a duality relation for anomalous dimensions. This functional equation defines a map between $\gamma = \gamma(\lambda, S)$ and $\gamma' = \gamma(\lambda, S')$ at complementary S, S' values of spins. These values of spins are related by:

$$\frac{S}{\sqrt{\lambda}}\frac{S'}{\sqrt{\lambda}} \approx \frac{1}{\pi}.$$

Thus the strings having spins S and S' are complementary-dual to each other:

$$S \gg \sqrt{\lambda} \gg 1 \qquad \Leftrightarrow \qquad \sqrt{\lambda} \gg S \gg 1,$$

that is in regions where the spin is much larger and much smaller than the coupling constant.

2 *Short-long strings duality*

An approximate description of the closed strings on the leading Regge trajectory is given by folded closed string which spins as a rigid rod around its center in the warped AdS_5 space background with the global metric

$$ds^2 = R^2(-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2). \tag{9}$$

A folded closed string whose center lies at $\rho = 0$ is spinning at the equatorial plane of S^3 and it is stretched from $\rho = 0$ to $\rho = \rho_0$. The polar angles are fixed, $\theta = \theta_1 = \pi/2$, the azimuthal angles ϕ depends on time $t = \tau$ and ρ is a function of σ

$$\phi = \omega t, \quad \rho = \rho(\sigma),$$

so that the energy and the spin of the string are [1]

$$E = \frac{4R^2}{2\pi\alpha'} \int_0^{\rho_0} \frac{\cosh^2 \rho d\rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}} \tag{10}$$

$$S = \frac{4R^2}{2\pi\alpha'} \int_0^{\rho_0} \frac{\omega \sinh^2 \rho d\rho}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}} \tag{11}$$

where ρ_0 is the maximal radial coordinate which depends on angular velocity

$$\tanh^2 \rho_0 = 1/\omega^2.$$

It follows from above formula that for large angular velocities $\omega \gg 1$

$$\rho_0 \sim 1/\omega$$

and the string is not stretched much compared to the radius of AdS_5, thus it is a "short string". When ω approaches 1 from above $\omega \sim 1$

$$\rho_0 \sim \frac{1}{2} \ln \frac{4}{1 - 1/\omega^2}$$

the string is much longer than the radius of curvature of AdS_5 and ρ_0 approaches the boundary of AdS_5 so that we have a "long string".

With the substitution $\omega \tanh \rho = sin\varphi$ for the energy (10) and spin (11) we shall get

$$E = \frac{4R^2}{2\pi\alpha' \omega} \int_0^{\pi/2} \frac{d\varphi}{(1 - \frac{1}{\omega^2} \sin^2 \varphi)^{3/2}} \tag{12}$$

$$S = \frac{4R^2}{2\pi\alpha'} [\int_0^{\pi/2} \frac{d\varphi}{(1 - \frac{1}{\omega^2} \sin^2 \varphi)^{3/2}} - \int_0^{\pi/2} \frac{d\varphi}{(1 - \frac{1}{\omega^2} \sin^2 \varphi)^{1/2}}] \tag{13}$$

and using the identity

$$\int_0^{\pi/2} \frac{d\varphi}{(1 - k^2 \sin^2 \varphi)^{3/2}} = \frac{1}{(1 - k^2)} \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{1/2} d\varphi \tag{14}$$

we can express the energy and the spin in terms of complete elliptic integrals of the first $\mathbf{K}(k)$ and second $\mathbf{E}(k)$ kinds

$$E = \frac{2\sqrt{\lambda}}{\pi} \frac{1}{\omega} \frac{1}{(1 - 1/\omega^2)} \mathbf{E}(1/\omega) \tag{15}$$

$$S = \frac{2\sqrt{\lambda}}{\pi} (\frac{1}{1 - 1/\omega^2} \mathbf{E}(1/\omega) - \mathbf{K}(1/\omega)), \tag{16}$$

where we use the map $R^2 = \sqrt{\lambda}\alpha'$ and that $k = 1/\omega$. These relations define $E(\lambda, \omega)$ and $S(\lambda, \omega)$ in a parametric form. Therefore in order to express the energy as a function of spin S one should invert the spin function $\omega = \omega(\lambda, S)$. It is a nontrivial problem and to resolve it we shall develop a special analytical tools in the next sections.

The well known Legendre relation between complementary elliptic integrals has the form

$$\mathbf{E}(k)\mathbf{K}(k') + \mathbf{K}(k)\mathbf{E}(k') - \mathbf{K}(k)\mathbf{K}(k') = \frac{\pi}{2} \tag{17}$$

where complementary module k' is defined as $k'^2 + k^2 = 1$. Using the above Legendre relation we shall get the duality relation

$$\frac{1}{\omega} ES' + \frac{1}{\omega'} E' S - SS' = \frac{2\lambda}{\pi} \tag{18}$$

where

$$\frac{1}{\omega^2} + \frac{1}{\omega'^2} = 1. \tag{19}$$

When $\omega \gg 1$ we have $\omega' \sim 1$ and vice-versa, therefore it defines exact map between energies and spins of *short and long strings*. Now it can be rewritten in terms of anomalous dimension as

$$\frac{1}{\omega} \gamma S' + \frac{1}{\omega'} \gamma' S + (\frac{1}{\omega} + \frac{1}{\omega'} - 1) SS' = \frac{2\lambda}{\pi} . \tag{20}$$

This functional equation defines the duality map between anomalous dimensions $\gamma = \gamma(\lambda, S)$ and $\gamma' = \gamma(\lambda, S')$ at complementary values of spins. Complementary values of spins are defined through the equation (19). The question is: What is omega large and omega small in terms of spins? In other words which values of S and S' are complementary? As we shall see in the next section, in the leading approximation $1 - \frac{1}{\omega^2} = x = \frac{2\sqrt{\lambda}}{\pi S}$ for large $S/\sqrt{\lambda} >> 1$, while $\frac{1}{\omega^2} = \frac{2S}{\sqrt{\lambda}}$ for small $S/\sqrt{\lambda} << 1$. Then equation (19) gives

$$\frac{S}{\sqrt{\lambda}}\frac{S'}{\sqrt{\lambda}} \approx \frac{1}{\pi}. \tag{21}$$

Thus the following regions are complementary-dual to each other:

$$S \gg \sqrt{\lambda} \gg 1 \qquad \Leftrightarrow \qquad \sqrt{\lambda} \gg S \gg 1, \tag{22}$$

that is the regions where the spin is much larger and much smaller than the coupling constant.

3 *Inverse Spin Function and Anomalous Dimension*

In order to study the behavior of the anomalous dimension in the subleading approximations one should have well defined expansion of complete elliptic integrals at two edges of the module space: $k = 1/\omega \sim 0$ and $k = 1/\omega \sim 1$. Using formulas (61) and (62) we can represent the energy and the spin (15) in the following form (see Appendix for details)

$$\begin{aligned} E = & \frac{4R^2}{2\pi\alpha'\omega}\{ \frac{1}{(1-1/\omega^2)} - \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!}(1-1/\omega^2)^n \times \\ \times & (\ln(1-1/\omega^2) + \psi(n+1/2) + \psi(n+3/2) - \psi(n+1) - \psi(n+2))\ \}, \end{aligned} \tag{23}$$

$$\begin{aligned} S = & \frac{4R^2}{2\pi\alpha'}\{ \frac{1}{(1-1/\omega^2)} - \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!}(1-1/\omega^2)^n \times \\ \times & (\ln(1-1/\omega^2) + \psi(n+1/2) + \psi(n+3/2) - \psi(n+1) - \psi(n+2)) + \\ + & \frac{1}{2\pi}\sum_{n=0}^{\infty}(\frac{\Gamma(n+1/2)}{n!})^2(1-1/\omega^2)^n \times \\ \times & (\ln(1-1/\omega^2) + 2\psi(n+1/2) - 2\psi(n+1))\ \}. \end{aligned} \tag{24}$$

The explicit expansion of the first few terms has the form

$$\begin{aligned} E(\omega) = & \frac{4R^2}{2\pi\alpha'\omega}\{ \frac{1}{(1-1/\omega^2)} - \frac{1}{4}[\ln(1-1/\omega^2) - 4\ln 2 + 1] - \\ & -\frac{3}{32}(1-1/\omega^2)[\ln(1-1/\omega^2) - 4\ln 2 + \frac{13}{6}] + ...\}, \end{aligned} \tag{25}$$

$$\begin{aligned} S(\omega) = & \frac{4R^2}{2\pi\alpha'}\{ \frac{1}{(1-1/\omega^2)} + \frac{1}{4}[\ln(1-1/\omega^2) - 4\ln 2 - 1] + \\ & + \frac{1}{32}(1-1/\omega^2)[\ln(1-1/\omega^2) - 4\ln 2 + \frac{3}{2}] + ...\}. \end{aligned} \tag{26}$$

We can represent now the energy and the spin (15) in the form convenient for expansion. The spin can be represented in the form

$$\mathcal{S} = \frac{\pi}{2}\frac{S}{\lambda^{1/2}} = \{\ \frac{1}{x} + \sum_{n=0}^{\infty} x^n (c_n \ln x + b_n)\} \tag{27}$$

and the energy as

$$\mathcal{E} = \frac{\pi}{2}\frac{E}{\lambda^{1/2}} = \sqrt{1-x}\ (\ \frac{1}{x} + \sum_{n=0}^{\infty} x^n (d_n \ln x + h_n)\ ) \tag{28}$$

where $x = 1 - 1/\omega^2$ and

$$\begin{aligned} d_n &= -\frac{1}{2^{2n+2}}\frac{(2n-1)!!(2n+1)!!}{n!(n+1)!}, \\ h_n &= d_n\ [\ \sum_{k=1}^{n}\frac{2}{k(2k-1)} + \frac{1}{(n+1)(2n+1)} - 4\ln 2] \\ c_n &= d_n + \frac{1}{2^{2n+1}}(\frac{(2n-1)!!}{n!})^2, \\ b_n &= h_n\ +\ \frac{1}{2^{2n+1}}(\frac{(2n-1)!!}{n!})^2[\ \sum_{k=1}^{n}\frac{2}{k(2k-1)} - 4\ln 2]. \end{aligned} \tag{29}$$

The explicit expression for the first few coefficients is

$$\begin{aligned} d_0 &= -\frac{1}{4}, & d_1 &= -\tfrac{3}{32}, & d_2 &= -\frac{15}{2\cdot 128}, \\ h_0 &= \ln 2 - \frac{1}{4}, & h_1 &= \tfrac{3}{8}\ln 2 - \tfrac{13}{64}, & h_2 &= \frac{15}{64}\ln 2 - \frac{9}{64}, \\ c_0 &= \frac{1}{4}, & c_1 &= \tfrac{1}{32}, & c_2 &= \frac{3}{2\cdot 128}, \\ b_0 &= -\ln 2 - \frac{1}{4}, & b_1 &= -\tfrac{1}{8}\ln 2 + \tfrac{3}{64}, & b_2 &= -\frac{6}{128}\ln 2 + \frac{3}{128}. \end{aligned} \tag{30}$$

In order to invert the spin function $x = x(\mathcal{S})$ we shall define the function $x^*(\mathcal{S})$ as a solution of the "reduced" spin equation (27)

$$\mathcal{S} = \frac{1}{x^*} + c_0 \ln x^* + b_0. \tag{31}$$

The benefit of using the function $x^*(\mathcal{S})$ is that the inverse spin function $x = x(\mathcal{S})$ can be expressed in terms of $x^*(\mathcal{S})$. The function $x^*(\mathcal{S})$ can be found by iteration of the following map (see Fig 1)

$$F(x) = \frac{1}{\mathcal{S} - c_0 \ln x - b_0}. \tag{32}$$

Indeed the iteration $x_n = FF....F(x_0)$ which starts from $x_0 = \frac{1}{\mathcal{S}}$ gives

$$x_0 = \frac{1}{\mathcal{S}} \rightarrow \frac{1}{\mathcal{S} - c_0 \ln 1/\mathcal{S} - b_0} \rightarrow \frac{1}{\mathcal{S} - c_0 \ln \frac{1}{\mathcal{S} - c_0 \ln 1/\mathcal{S} - b_0} - b_0} \rightarrow ... \rightarrow x^*(\mathcal{S}). \tag{33}$$

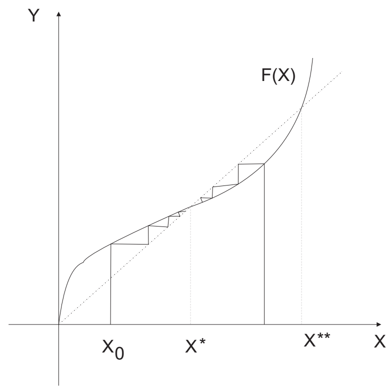

Figure 1: The iteration function $F(x) = 1/(\mathcal{S} - c_0 \ln x - b_0)$ has two nontrivial fixed points x^* and x^{**}, which are the solutions of the equation $F(x) = x$. The iteration is defined as $x_n = FF....F(x_0)$ and converges to the stable fixed point $x_n \to x^*$. In the given case the fixed point x^* is a stable one, while the second one x^{**} is unstable and unphysical.

and has the property that its fixed point $F(x^*) = x^*$ is a solution of the equation (31). The iteration process can be represented in the form of infinite product

$$
\begin{aligned}
x_0 &= \frac{1}{\mathcal{S}} \\
x_1 &= \frac{1}{\mathcal{S}} \cdot \frac{1}{1 - A/\mathcal{S}} \\
x_2 &= \frac{1}{\mathcal{S}} \cdot \frac{1}{1 - A/\mathcal{S}} \cdot \frac{1}{1 + c_0 \frac{1}{\mathcal{S}} \frac{1}{(1-A/\mathcal{S})} \ln(1 - A/\mathcal{S})} \\
&\cdots\cdots\cdots\cdots \\
x^* &= \frac{1}{\mathcal{S}} \cdot \frac{1}{1 - A/\mathcal{S}} \cdot \frac{1}{1 + c_0 \frac{1}{\mathcal{S}} \frac{\ln(1-A/\mathcal{S})}{(1-A/\mathcal{S})}} \cdot \frac{1}{1 + c_0 \frac{1}{\mathcal{S}} \frac{1}{1-A/\mathcal{S}} \frac{1}{(1+\frac{c_0}{\mathcal{S}} \frac{\ln(1-A/\mathcal{S})}{(1-A/\mathcal{S})})} \ln(1 + \frac{c_0}{\mathcal{S}} \frac{\ln(1-A/\mathcal{S})}{(1-A/\mathcal{S})})} \cdots
\end{aligned}
\tag{34}
$$

where

$$
A = c_0 \ln(1/\mathcal{S}) + b_0 = -\frac{1}{4}(\ln 16\mathcal{S} + 1). \tag{35}
$$

It follows therefore that the fixed point solution $x^*(\mathcal{S})$ can be represented in the form of infinite product in which x_1/x_0 contains the leading terms $\frac{\ln^n \mathcal{S}}{\mathcal{S}^n}$, the x_2/x_1 contains subleading terms $\frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+1}}$, the x_3/x_2 contains next to subleading terms $\frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+2}}$ and so on. In general, each subsequent k-th term in the infinite product gives the higher orders terms in the expansion over $\frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+k}}$. The above infinite product representation (34) of the fixed point solution $x^*(\mathcal{S})$ allows also the representation in the form of infinite sum

$$
x^*(\mathcal{S}) = \frac{1}{\mathcal{S}} \cdot [1 + X_1\left(\frac{A}{\mathcal{S}}\right)] \cdot [1 + \frac{1}{\mathcal{S}} X_2(\frac{A}{\mathcal{S}})] \cdot .... = \frac{1 + X_1}{\mathcal{S}} + \frac{(1 + X_1)X_2}{\mathcal{S}^2} + ... \tag{36}
$$

The infinite product representation (34) can be conveniently used when one should compute the logarithm of x^* and the infinite sum representation (36) when one should compute polynomials of x^*.

The advantage of using the fixed point solution of the reduced spin equation is that we can find the inverse spin function $x = x(\mathcal{S})$ in any required approximation using the

fixed point solution $x^*(\mathcal{S})$. The inverse spin function $x(\mathcal{S})$ can be represented as a sum of the fixed point solution $x^*(\mathcal{S})$ plus terms which are next to subleading and higher

$$x(\mathcal{S}) = x^*(\mathcal{S}) + \sum_{n=0}^{\infty} a_n \frac{(\ln 1/\mathcal{S})^n}{\mathcal{S}^{n+2}} + ... = x^* + \delta x_2 + ... \tag{37}$$

Substituting the expansion (37) into (27) we shall get the next to subleading term $\delta_2 x$ as a function of x^*

$$\delta x_2 = c_1 (x^*)^3 \ln x^*. \tag{38}$$

In order to find leading and subleading terms in the anomalous dimension we have to keep quadratic in x terms in the $\mathcal{E} - \mathcal{S}$ and use the relations which follow from (37) and (38). Indeed the next to subleading term δx_2 in $x(S)$ will generate the following subleading terms when multiplied by the logarithm of x^* or divided by x^*

$$\delta x_2 \;\; \ln x^* \quad \to \quad \sum_{n=0}^{\infty} a_n \frac{(\ln 1/\mathcal{S})^{n+1}}{\mathcal{S}^{n+2}}, \qquad \frac{\delta x_2}{x^*} \quad \to \quad \sum_{n=0}^{\infty} a_n \frac{(\ln 1/\mathcal{S})^n}{\mathcal{S}^{n+1}} \tag{39}$$

therefore we shall get

$$\begin{aligned}
\mathcal{E} - \mathcal{S} = (d_0 - c_0) \ln x + (h_0 - b_0 - \frac{1}{2}) + (d_1 - c_1 - \frac{1}{2} d_0) x \ln x + (h_1 - b_1 - \frac{1}{2} h_0 - \frac{1}{8}) x + \\
+ (d_2 - c_2 - \frac{1}{2} d_1 - \frac{1}{8} d_0) x^2 \ln x + (h_2 - b_2 - \frac{1}{2} h_1 - \frac{1}{8} h_0 - \frac{1}{16}) x^2 = \\
(d_0 - c_0) \ln x^* + (d_0 - c_0) \frac{\delta_2 x}{x^*} + (h_0 - b_0 - \frac{1}{2}) + (h_1 - b_1 - \frac{1}{2} h_0 - \frac{1}{8}) x^* + \\
+ (d_2 - c_2 - \frac{1}{2} d_1 - \frac{1}{8} d_0)(x^*)^2 \ln x^* + (h_2 - b_2 - \frac{1}{2} h_1 - \frac{1}{8} h_0 - \frac{1}{16})(x^*)^2 = \\
= (d_0 - c_0) \ln 1/\mathcal{S} + (h_0 - b_0 - \frac{1}{2}) + (d_0 - c_0) \sum_{n=1}^{\infty} \frac{(c_0 \ln 1/\mathcal{S} + b_0)^n}{n \mathcal{S}^n} + \\
+ (d_0 - c_0) c_0 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n\mathcal{S}} \left(\frac{c_0 \ln 1/\mathcal{S}}{\mathcal{S}} \right)^{n+k-1} + \\
+ (d_0 - c_0) c_1 \sum_{n=1}^{\infty} n \frac{(c_0 \ln 1/\mathcal{S} + b_0)^{n-1} \ln 1/\mathcal{S}}{\mathcal{S}^{n+1}} \\
+ (h_1 - b_1 - \frac{1}{2} h_0 - \frac{1}{8}) \frac{1}{\mathcal{S}} + (h_1 - b_1 - \frac{1}{2} h_0 - \frac{1}{8}) \sum_{n=1}^{\infty} \frac{(c_0 \ln 1/\mathcal{S} + b_0)^n}{\mathcal{S}^{n+1}} + \\
+ (d_2 - c_2 - \frac{1}{2} d_1 - \frac{1}{8} d_0) \sum_{n=1}^{\infty} n \frac{(c_0 \ln 1/\mathcal{S} + b_0)^{n-1} \ln 1/\mathcal{S}}{\mathcal{S}^{n+1}} + \\
+ (h_2 - b_2 - \frac{1}{2} h_1 - \frac{1}{8} h_0 - \frac{1}{16}) \frac{1}{\mathcal{S}^2} + ... \mathcal{O}\left(\frac{\ln^m 1/\mathcal{S}}{\mathcal{S}^{m+2}} \right) \quad .
\end{aligned} \tag{40}$$

Representing this expansion in the following from

$$\mathcal{E} - \mathcal{S} = \frac{1}{2} \ln \mathcal{S} + \rho_0 + \sum_{n=1}^{\infty} \rho_{nn} \frac{\ln^n \mathcal{S}}{\mathcal{S}^n} + \sum_{n=1}^{\infty} \rho_{nn-1} \frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+1}} + \rho_1 \frac{1}{\mathcal{S}} + \rho_2 \frac{1}{\mathcal{S}^2} \tag{41}$$

we can derive all these coefficients

$$\rho_0 \;\; = \;\; h_0 - b_0 - \frac{1}{2} = 2 \ln 2 - \frac{1}{2}$$

$$
\begin{aligned}
\rho_{nn} &= \frac{(-1)^n}{n}(d_0-c_0)c_0^n = \frac{1}{2}\frac{(-1)^{n+1}}{4^n}\frac{1}{n} \\
\rho_{nn-1} &= (-1)^n c_0^n \ [\ (d_0-c_0)b_0 + (d_0-c_0)c_0\sum_{k=1}^{n}\frac{1}{k} + h_1 - b_1 - \frac{1}{2}h_0 - \frac{1}{8}\ ]\ + \\
&\quad +(-1)^n n\ c_0^{n-1}\ [\ (d_0-c_0)c_1 + d_2 - c_2 - \frac{1}{2}d_1 - \frac{1}{8}d_0\ ] \\
&= \frac{1}{2}\frac{(-1)^{n+1}}{4^n}\ (\frac{n+4}{16} + \frac{1}{4}\sum_{k=1}^{n}\frac{1}{k} - \ln 2) \\
\rho_1 &= h_1 - b_1 - \frac{1}{2}h_0 - \frac{1}{8} + (d_0-c_0)b_0 = \frac{1}{2}\ln 2 - \frac{1}{8} \\
\rho_2 &= (d_0-c_0)(c_0b_0 + b_0^2/2) + (h_1 - b_1 - \frac{1}{2}h_0 - \frac{1}{8})b_0 + \\
&+\ h_2 - b_2 - \frac{1}{2}h_1 - \frac{1}{8}h_0 - \frac{1}{16} = -\frac{1}{4}\ln^2 2 + \frac{7}{32}\ln 2 - \frac{1}{64}.
\end{aligned}
\tag{42}
$$

Finally the dependence of the anomalous dimension from the spin will take the form

$$
E - S = \frac{\sqrt{\lambda}}{\pi}\ln(S/\sqrt{\lambda}) + f_0 + \sum_{n=1}^{\infty} f_{nn}\frac{\ln^n(S/\sqrt{\lambda})}{S^n} + \sum_{n=1}^{\infty} f_{nn-1}\frac{\ln^n(S/\sqrt{\lambda})}{S^{n+1}} + f_1\frac{1}{S} + f_2\frac{1}{S^2} + \ldots
\tag{43}
$$

where

$$
\begin{aligned}
f &= \frac{\sqrt{\lambda}}{\pi} \\
f_0 &= \frac{\sqrt{\lambda}}{\pi}\ (2\rho_0 + \ln(\pi/2)) = \frac{\sqrt{\lambda}}{\pi}(\ln 8\pi - 1) \\
f_1 &= \frac{4\lambda}{\pi^2}\ (\rho_1 +\ \rho_{11}\ln(\pi/2)) = \frac{\lambda}{2\pi^2}(\ln 8\pi - 1) \\
f_2 &= \frac{8\lambda^{3/2}}{\pi^3}\ (\rho_2 + \rho_{10}\ln(\pi/2) + \rho_{22}\ln^2(\pi/2)) = \\
&= \frac{\lambda^{3/2}}{8\pi^3}(-\ln^2 8\pi + \frac{9}{2}\ln 8\pi - 4\ln 2 - 1) \\
f_{nn} &= (\frac{2\lambda^{1/2}}{\pi})^{n+1}\ \rho_{nn} = (-1)^{n+1}\frac{\lambda^{\frac{n+1}{2}}}{(2\pi)^{n+1}}\frac{2}{n} = \frac{(-1)^{n+1}}{2^n n}f^{n+1}, \qquad n = 1, 2, \ldots \\
f_{nn-1} &= (\frac{2\lambda^{1/2}}{\pi})^{n+2}\ (\ \rho_{nn-1} + \rho_{n+1n+1}(n+1)\ln(\pi/2)\ ) = \\
&= (-1)^{n+1}\frac{\lambda^{\frac{n+2}{2}}}{(2\pi)^{n+2}}\ (\ \frac{n+4}{2} + 2\sum_{k=1}^{n}\frac{1}{k} - 2\ln 8\pi\ ), \qquad n = 1, 2, \ldots
\end{aligned}
\tag{44}
$$

and

$$
\sqrt{\lambda} \gg 1, \qquad \frac{S}{\sqrt{\lambda}} \gg 1.
\tag{45}
$$

We close this Section by noticing that the iteration procedure described above can be generalised to give us all the terms in the large S expansion of the strong coupling anomalous dimension. For example, if one wanted to compute the infinite series of the coefficients f_{nn-2} in front of the sub-subleading terms $\frac{\ln^n S}{S^{n+2}}$ in (43) one should define an $x^*(\mathcal{S})$ which

is the solution of the equation

$$\mathcal{S} = \frac{1}{x^*} + c_0 \ln x^* + b_0 + x^*(c_1 \ln x^* + b_1). \tag{46}$$

Then the inverse spin function $x(\mathcal{S})$ can be represented as a sum of the fixed point solution $x^*(\mathcal{S})$ plus terms which are next to sub-subleading and higher

$$x(\mathcal{S}) = x^*(\mathcal{S}) + \sum_{n=0}^{\infty} t_n \frac{(\ln 1/\mathcal{S})^n}{\mathcal{S}^{n+3}} + ... = x^* + \delta x_3 + ... \tag{47}$$

δx_3 and as a result the coefficients t_n can be determined by demanding that (47) satisfies (27) up to subleading order, i.e. $\frac{\ln^n S}{S^{n+1}}$. More precisely one gets,

$$\delta x_3 = c_2 \ln x^* \; x^{*4}, \tag{48}$$

where x^* is the solution of (46). Then one can substitute the solution (47) in the expression for the energy minus spin, $E(x) - S(x)$, to get the anomalous dimension up to sub-subleading order, that is up to order $\frac{\ln^n S}{S^{n+2}}$.

Let us only make two observations. The first is that since we are interested in the sub-subleading terms we should keep terms which mostly go as $x^3(\ln x)$ or x^3 in the expansion for $\mathcal{E} - \mathcal{S}$ (see equations (27), (28)). The second is that since x^* is found by iteration of the following map

$$F(x) = \frac{1}{\mathcal{S} - c_0 \ln x - b_0 - x(c_1 \ln x + b_1)} \tag{49}$$

one should make as many iterations needed for reaching the sub-subleading order in the expression for $E - S$.

4 *Functional and Reciprocity Relation*

As we mentioned in the Introduction, the operators dual to the string states we are considering are of the form

$$\mathcal{O}_s = tr(Z D_+^S Z) + ... \tag{50}$$

and as such they belong to a $SL(2,R)$ subgroup of the full $PSU(2,2/4)$ group of $N=4$ SYM. They are labelled by the conformal spin $s = \frac{1}{2}(S+\Delta)$. Thus, it is natural to assume that the anomalous dimension γ of these operators depends on S only through the conformal spin $s = 1 + S + \frac{1}{2}\gamma$. This assumption leads to the following functional relation for γ [27, 28]

$$\gamma(S) = P(S + \frac{1}{2}\gamma(S)). \tag{51}$$

(51) can be inverted to give us a functional relation for P. Namely,

$$P(S) = \gamma(S - \frac{1}{2}P(S)). \tag{52}$$

Assuming that the anomalous dimension γ is known up to some order one can determine the function P and vice-versa. This can be achieved using the following relation

$$P(S) = \sum_{k=1}^{\infty} \frac{1}{k!}(-\frac{1}{2}\partial_S)^{k-1}\gamma^k(S). \tag{53}$$

Using the infinite series for f_{nn} given in (44) we see that the coefficients of the leading logarithms $\frac{\ln^n S}{S^n}$ in the expansion of $\gamma(S)$ are fully determined by the scaling function f. This is consistent with the functional relation (51) and one can rewrite $\gamma(S)$ as

$$\gamma(S) = \frac{\sqrt{\lambda}}{\pi}\ln(S/\sqrt{\lambda}) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n n}f^{n+1}\frac{\ln^n(S/\sqrt{\lambda})}{S^n} + ... = f\ln(S + \frac{1}{2}f\ln S + ...) + ... \tag{54}$$

From (54) it is obvious that the functional relation (51) holds for the leading logarithmic terms of the form $\frac{\ln^n S}{S^n}$, with P given by

$$P(S) = f\ln S + ..., \tag{55}$$

where the dots stand for subleading terms in S.

Another important consequence of the form of f_{nn}'s is that the function P obeys the so called "simplicity" condition [3] [34], which states that P is simpler than γ in the sense that it contains no leading logarithms in its expansion

$$P(S) = \frac{\sqrt{\lambda}}{\pi}\ln(S/\sqrt{\lambda}) + \hat{P}_0 + \sum_{n=1}^{\infty}\hat{P}_{nn-1}\frac{\ln^n(S/\sqrt{\lambda})}{S^{n+1}} + ... \tag{56}$$

Should P contained those terms the value of the coefficients f_{nn} would be different from that of (44).

Furthermore, it is conjectured that $P(S)$ satisfies a "parity preserving" or "reciprocity" relation. This kind of relation was first observed in the context of deep inelastic scattering (DIS) in QCD [20, 21, 25, 26]. It is a relation involving the splitting functions $P_s = P_t = P(x)$

$$P(x) = -xP\Big(\frac{1}{x}\Big). \tag{57}$$

These splitting functions can be related to the anomalous dimension of twist two operators through a Mellin transform. In the case of the maximally supersymmetric theory it can be cast in the form [27]

$$P(S) = \sum_{k=0}^{\infty}\frac{c_k(\ln C)}{C^{2k}}. \tag{58}$$

In (58), C is the "bare" quadratic Casimir operator of the $SL(2,R)$ group given by $C = s_0(s_0 - 1)$, where $s_0 = \frac{1}{2}(S + \Delta_0) = S + 1$ is the value of the conformal spin at the classical level. As a result, $C = S(S+1)$. Equation (58) implies relations between the subleading terms in the expansion (43). These relations can be obtained by comparing the coefficients of the subleading logs in equation (52). More precisely, by equating (56) and (58) it is immediate to get that $P_{2n\,2n-1} = 0$. Then, one can exploit (53) and compare the coefficients of the subleading logs to an even power, i.e. $\frac{\ln^{2n} S}{S^{2n+1}}$ to get the aforementioned relations between the subleading terms in the expansion (43). The comparison of the odd terms $\frac{\ln^{2n+1} S}{S^{2n+2}}$ determine the value of the non-zero coefficients $\hat{P}_{2n+1\,2n}$ of the subleading terms appearing in the expansion of $P(S)$ (56).

[3] We should note that this "simplicity" condition is broken in perturbation theory for twist two operators [32] and twist three operators [33] at critical wrapping order. We thank Valentina Forini for pointing this out.

The reciprocity relation (58) has been verified to hold in perturbation theory up to five loops for the anomalous dimension of twist two operators in $N=4$ SYM (see [34] and references therein). At strong coupling the first two MVV relations

$$
\begin{aligned}
f_1 &= \frac{1}{2}f(f_0+1) \\
f_{21} &= \frac{1}{16}f[f^3-2f^2(f_0+1)-16f_{10}] \\
&\;\;. \\
&\;\;. \\
&\;\;.
\end{aligned} \tag{59}
$$

were verified to hold [23]. Since now we have the infinite series of the coefficients of the subleading logs in the strong coupling expansion (43) it is possible to check if the relations following from the reciprocity property of the function P hold at any level. Here we focus on the next relation in the sequence of (59). As mentioned above, this can be derived by equating the coefficients multiplying the $\frac{\ln^4 S}{S^5}$ term in (58). To this end,we have to decide how many terms we should keep on the right hand side of (58). By taking 5 derivatives of a generic term $\frac{\ln^a S}{s^b}$ appearing in the right hand side of (58) we take terms like $\frac{\ln^{a-i} S}{S^{b+5}}$, where $i=0,1,...,5$. For these terms to be equal to $\frac{\ln^4 S}{S^5}$ we have to demand that $b=0$ and $a=9,8,7,6,5,4$. Notice that if we differentiate once more, take 6 derivatives of $\frac{\ln^a S}{S^b}$ then the denominator of the result would be S^{b+6} which implies that $b=-1$. But this is impossible since only negative powers of S appear in the expansion of the anomalous dimension or any positive power of it. As a result the last term which can be contribute to the coefficient of $\frac{\ln^4 S}{S^5}$ in the right hand side of (58) is the 5^{th} derivative of $\gamma^6(S)$.

By isolating the appropriate terms we get the next relation in the sequence (59). This reads

$$
\begin{aligned}
f_{43} + [f_{22}^2 + 2f_{11}f_{33} + 2ff_{32} + f_{44}(2f_0 - \frac{5}{2}f)] + f[3f_{11}f_{22} + \frac{3}{2}ff_{21} + f_{33}(-\frac{35}{8}f + 3f_0)] \\
+f^2[\frac{3}{4}f_{11}^2 + \frac{1}{2}ff_{10} + f_{22}(-\frac{65}{24}f + \frac{3}{2}f_0)] + f^3[\frac{1}{16}ff_1 + f_{11}(-\frac{125}{12\;16}f + \frac{1}{4}f_0)] \\
+f^5[\frac{25}{12\;32}f - \frac{1}{32}f_0] = 0.
\end{aligned} \tag{60}
$$

Now we can use (44) to find and substitute the values for the various coefficients needed in (60) to find that this equation is, indeed, satisfied. We conclude that all the constraints following from the reciprocity relation (58) are likely to be satisfied in string perturbation theory too.

5 Acknowledgements

We would like to thank E. Floratos for stimulating discussions and Valentina Forini for useful comments on the first version of this work.

6 *Appendix*

For the small values of the module $|k^2| < 1$ the elliptic integral of the first kind $\mathbf{K}(k)$ has the well known expansion

$$\mathbf{K}(k) = \frac{\pi}{2}\{1 + \sum_{n=1}^{\infty}[\frac{(2n-1)!!}{2n!!}]^2 \cdot k^{2n} \ \},\qquad |k^2| < 1$$

and for the large values $|1 - k^2| < 1$ we have found the following expansion

$$\mathbf{K}(k) = -\frac{1}{2\pi}\sum_{n=0}^{\infty}(\frac{\Gamma(n+1/2)}{n!})^2(1-k^2)^n \cdot [\ \ln(1-k^2) + 2\psi(n+1/2) - 2\psi(n+1)\ ], \quad (61)$$

where $\psi(z)$ is the digamma function. For the elliptic integral of the second kind $\mathbf{E}(k)$ the expansion for small values of the module $|k^2| < 1$ is

$$\mathbf{E}(k) = \frac{\pi}{2}\{1 - \sum_{n=1}^{\infty}[\frac{(2n-1)!!}{2n!!}]^2 \cdot \frac{k^{2n}}{2n-1}\},\qquad |k^2| < 1$$

and for the large values $|1 - k^2| < 1$ it is

$$\begin{aligned} \mathbf{E}(k) &= 1 - \frac{(1-k^2)}{2\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!}(1-k^2)^n \cdot \\ &\cdot\ [\ \ln(1-k^2) + \psi(n+1/2) + \psi(n+3/2) - \psi(n+1) - \psi(n+2)\ ]. \quad (62) \end{aligned}$$

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